Sheaves

(See Har II.1, Shaf V.2)

In order to finish defining schemes, we need to define sheaves, which are an important part of the data of a scheme. We first define sheaves very generally for any topological space, and then in the next section give the important example of the structure sheaf on a scheme.

Motivating example: differentiable functions

A natural example that you have all seen (in some capacity) is the sheaf of differentiable functions on a manifold, or even just on $\mathbb{R}$:

On each open set U, let $\mathcal{D}(U)$ be the group of differentiable functions $f: U \to \mathbb{R}$.

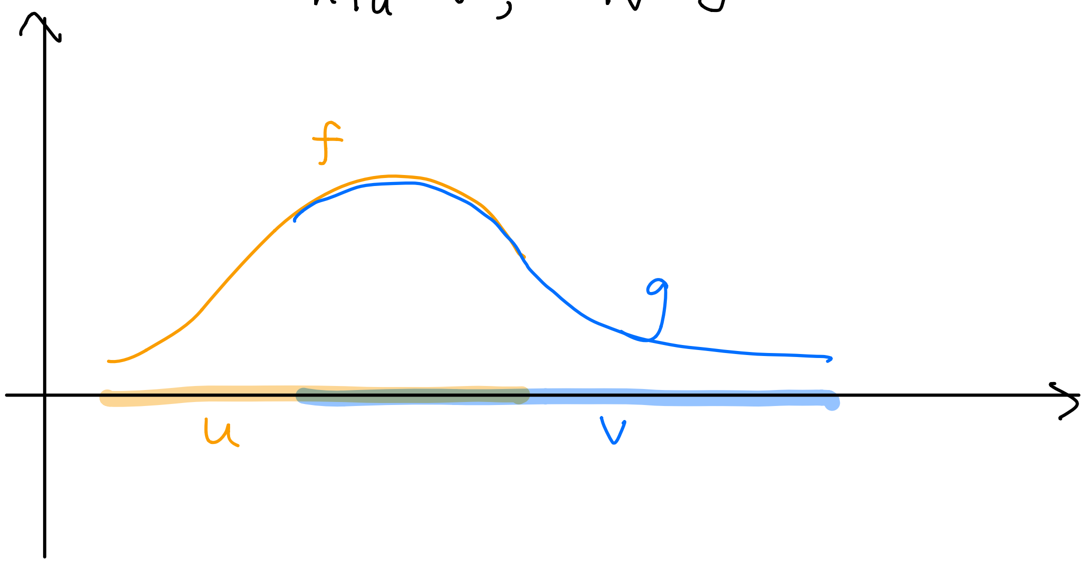
So, e.g. $f(x) = 1/x \in \mathcal{D}(\mathbb{R} - \{0\})$ but not in $\mathcal{D}(\mathbb{R})$.

For $U \subseteq V$ open sets, there is a natural group homomorphism $\mathcal{D}(V) \to \mathcal{D}(U)$ given by the restriction $f \mapsto f|_U$.

Note that these functions can be "glued" together as follows:

If $f \in \mathcal{D}(U)$, $g \in \mathcal{D}(V)$ and $f|_{U\cap V} = g|_{U \cap V}$, there is a unique $h \in \mathcal{D}(U \cup V)$ such that

$$h|_U = f, \quad h|_V = g.$$



We can explicitly define it as

$$h(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$

This is a good example to keep in mind as we give the following general (and almost inscrutably abstract) definitions.

For the remainder of this section, let X be a topological space.

Def: A <u>presheaf</u> $\mathcal{F}$ of abelian groups on X consists of the following.

1.) for every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$, called the <u>sections</u> over U, and

2.) for every inclusion $V \subseteq U$ of open sets of X, a group homomorphism

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V), \ *$$

called a <u>restriction map</u>,

which satisfy the following:

$*$Note: If $s \in \mathcal{F}(U)$, we write $s|_V$ to denote $\rho_{UV}(s)$

- $\mathcal{F}(\phi) = 0$,
- $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map, and
- if $W \subseteq V \subseteq U$ are open sets, then

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV},$$

i.e. the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[d, "\rho_{VW}"] \\ & \mathcal{F}(W) \end{tikzcd}.$$

This gives a def of a presheaf of abelian groups, but you can replace abelian groups w/ rings, sets, or objects from any other fixed category.

A sheaf is a presheaf where the sections are determined by "local data". More precisely:

Def: A presheaf $\mathcal{F}$ on X is a <u>sheaf</u> if it also satisfies the following:

Let $U \subseteq X$ be an open set and $\{V_i\}$ an open covering of U

1.) If $s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = 0$ $\forall$ i, then $s = 0$, and

2.) if $s_i \in \mathcal{F}(V_i)$ for each i such that

$$s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j} \text{ for each } i, j,$$

then there is an element $s \in \mathcal{F}(U)$ s.t. $s|_{V_i} = s_i$ for each i.

Note: 1.) implies s in 2.) is unique:

suppose s and t both work. Then

$s|_{V_i} = t|_{V_i} = s_i$ for each i, so $(s-t)|_{V_i} = 0$ for each i,

so $s - t = 0 \Rightarrow s = t$.

Remark: One of the perks of the additional conditions required for sheaves is that it allows us to uniquely describe a sheaf by only giving the sections on a basis.

(Recall: Given a topology on X, a basis is a subset of the open sets s.t. for all $U \subseteq X$ open, U is the union of elements of the basis.)

Example: Regular functions on an (affine) variety.

Let $X \subseteq \mathbb{A}^n$ be an affine variety defined $X = V(I)$, I some radical ideal.

For each open set $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions $U \to k$, and for $V \subseteq U$, $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ the restriction map.

Check: this is a sheaf, called the <u>sheaf of regular functions</u> on X.

Notice: $\mathcal{O}(X) = k[x_1, \dots, x_n]/I = \Gamma(X)$.

What is $\mathcal{O}(U)$ if $U = X \setminus V(f)$?

Each $\mathcal{O}(U) \subseteq k(X)$, the field of rational functions on X (which in this case is just the field of fractions of $\Gamma(X)$).

$\frac{g}{h}$ is regular on U as long as h only vanishes <u>away</u> from U. i.e.

$$\mathcal{O}(U) = \Gamma(X)\left[\tfrac{1}{f}\right]. \quad \text{(check this!)}$$

(Can you figure out what $\mathcal{O}(U)$ is for $U = X \setminus V(I)$?)

We will see that the structure sheaf on Spec$R$ generalizes this — more details in the next section.

Ex: Constant functions. Consider X a topological space and $\mathcal{F}(U) = \mathbb{Z}$ for every $U \subseteq X$ open, except $\mathcal{F}(\emptyset) = 0$.

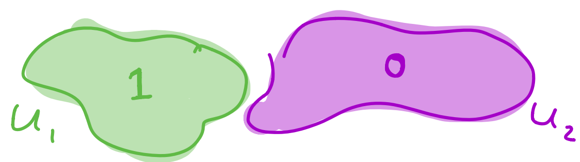
For $\emptyset \neq V \subseteq U$, define $\rho_{UV} = \mathrm{id}_{\mathbb{Z}}$. This is a presheaf, but it is not a sheaf:

Let U_1, U_2 be disjoint nonempty open sets.

Let $s_1 = 1 \in \mathcal{F}(U_1)$, and $s_2 = 0 \in \mathcal{F}(U_2)$.

Then $s_1|_{U_1 \cap U_2} = 0 = s_2|_{U_1 \cap U_2}$, since $U_1 \cap U_2 = \emptyset$.

However, there is no section over $U_1 \cup U_2$ that restricts to 1 on U_1 and 0 on U_2.

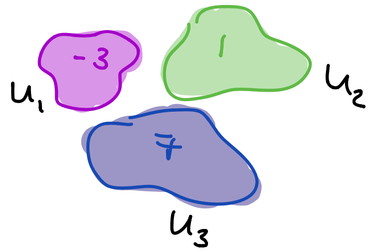


We can "fix" this as follows:

Define the sheaf of locally constant functions $\mathcal{F}'$ as $\mathcal{F}'(\emptyset) = 0$, and for $\emptyset \neq U \subseteq X$ open,

$$\mathcal{F}'(U) = \left\{ f : U \to \mathbb{Z} \;\middle|\; f(x) = f(y) \text{ if } x, y \text{ in same connected comp. of } U \right\}$$

and the restriction maps are just the usual restrictions of functions.



Then $\mathcal{F}'(U) = \prod_{i \in I} \mathbb{Z}$,
where I indexes the connected components of U. (Check: $\mathcal{F}'$ is a sheaf.)

Direct limits and stalks

If $\mathcal{F}$ is a presheaf on X and $P \in X$, the stalk of $\mathcal{F}$ at P captures the behavior of $\mathcal{F}$ "near" P.

How do we do this? Look at sections of open sets containing P that agree on the intersection. To make this precise, we need the direct limit.

Def: The stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P is the direct limit of the $\mathcal{F}(U)$ for all open sets $U \subseteq X$ containing P.
That is,

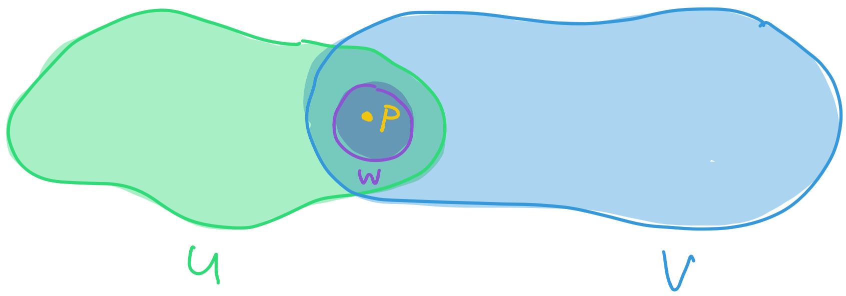
$$\mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U)$$

$$:= \{(U, f \in \mathcal{F}(U))\}/\sim$$

where

$$(U, f \in \mathcal{F}(U)) \sim (V, g \in \mathcal{F}(V))$$

if there is an open $W \subseteq U \cap V$ containing P s.t. $f|_W = g|_W$.



So an element $s_p \in \mathcal{F}_p$ can be <u>represented</u> by a section $s \in \mathcal{F}(U)$ for some $U \ni P$. We call s_p the <u>germ</u> of the section s at P.

<u>Ex</u>: Consider the sheaf of regular functions $\mathcal{O}$ on $\mathbb{A}^2$, as described above. Let $P = (0,0)$. What is $\mathcal{O}_p$?

Since $\mathcal{O}(U) \subseteq k(\mathbb{A}^2)$ for each U, we have

$\mathcal{O}_p \subseteq k(\mathbb{A}^2)$ as well.

Notice: $\frac{f}{g} \in \mathcal{O}_p \Leftrightarrow$ it is regular in a neighborhood of P

$\Leftrightarrow$ it is regular at P
(check!)

So the nonunits are of the form $\frac{f_1 x + f_2 y}{g}$, where $g \notin (x,y)$.

$\Rightarrow$ The nonunits form an ideal (x,y), so $\mathcal{O}_p$ is a local ring w/ max'l ideal (x,y).

(Exercise from CA if you've never done it:
R is local $\Leftrightarrow$ nonunits form an ideal)